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SUBORDINATION FAMILIES AND EXTREME POINTS

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ABSTRACT. Let $s(F)$ denote the set of functions subordinate to a univalent function F in Δ the unit disk. Let B_0 denote the set of functions $\phi(z)$ analytic in Δ satisfying $|\phi(z)| < 1$ and $\phi(0) = 0$. We prove that if $f = F \circ \phi$ is an extreme point of $s(F)$, then ϕ is an extreme point of B_0. Let $D = F(s)$ and $\lambda(w, \partial D)$ denote the distance between w and ∂D (boundary of D). We also prove that if ϕ is an extreme point of B_0 and $|\phi(e^{it})| < 1$ for almost all t, then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D)\, dt = -\infty$ for almost all θ.

Introduction. Let $\Delta = \{z: |z| < 1\}$ and let $\mathbf{A}$ denote the set of functions analytic in Δ. Let B_0 denote the subset of $\mathbf{A}$ consisting of all functions ϕ that satisfy the conditions $|\phi(z)| < 1$, $\phi(0) = 0$. Let EB_0 denote the extreme points of B_0. Let S denote the subset of $\mathbf{A}$ consisting of univalent functions f so that $f(z) = z + \cdots$ in Δ.

Let F be in $\mathbf{A}$ and be univalent in Δ. Let $s(F)$ denote the subset of $\mathbf{A}$ consisting of functions f that are subordinate to F in Δ. This means that $f \in \mathbf{A}$, $f(0) = F(0)$, and $f(\Delta) \subset F(\Delta)$. These conditions are equivalent to the existence of $\phi \in B_0$ so that $f = F \circ \phi$. Note that $s(F) = \{F \circ \phi: \phi \in B_0\}$.

Let D denote $F(\Delta)$. It is known that $F \in H^p$ for all $p < 1/2$ [**3**, p. 50] and so if $f = F \circ \phi$ for $\phi \in B_0$, then $f \in H^p$ for $p < 1/2$ [**3**, pp. 10–11]. It follows that $\lim_{r \to 1} f(re^{i\theta}) = f(e^{i\theta})$ exists almost everywhere.

We let $Es(F)$ denote the set of extreme points of $s(F)$. In [**1**] it was proved that if F' is in the Nevanlinna class and D is a Jordan domain subset to a half plane, then $Es(F) \subset \{F \circ \phi: \phi \in EB_0\}$. In that paper it was conjectured that this inclusion is valid for any univalent F. In [**4**] the inclusion was proved under the assumption F is univalent and D is a Jordan domain. In this paper we prove the inclusion for an arbitrary univalent function and so verify the conjecture made by the first author in [**1**].

In [**1**] it was also proved that if F' is in the Nevanlinna class and $\phi \in EB_0$, then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty$. It was conjectured that the integral was $-\infty$ for any univalent function F when $\phi \in EB_0$. (Note that this is trivially true if $|\phi(e^{it})| = 1$ on a set of positive measure since F is univalent.) A weaker conjecture is that if $F \circ \phi \in Es(F)$ and F is univalent, then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty$.

In this paper we show that this weaker conjecture holds when F' is in the Nevanlinna class [**3**, p. 16]. We also take this opportunity to point out that in the proof

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of Theorem 8.25 in [**5**, pp. 143–144] there is a proof of this weaker conjecture when $F(\Delta)$ is a Jordan domain and no assumption on F' is made. Also it is known that $\int_0^{2\pi} \log \lambda(f(\phi(e^{it})), \partial D)\, dt = -\infty$ does not in general imply $F \circ \phi \in Es(F)$. This can be easily seen by considering the case $F(z) = ((1+z)/(1-z))^\alpha$ for $0 < \alpha < 1$ [**5**, pp. 131, 133]. We prove the interesting result that if F is univalent and $\phi \in EB_0$, then $\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D)\, dt = -\infty$ for almost all θ.

Functions subordinate to a univalent function.

THEOREM 1. *If F is a univalent function analytic in Δ and $f = F \circ \phi$ is an extreme point of $s(F)$, then ϕ is an extreme point of B_0.*

PROOF. Assume $f = F\circ\phi \in Es(F)$ and $\phi \notin EB_0$. Then $\int_0^{2\pi} \log(1-|\phi(e^{it})|)\, dt > -\infty$ [**3**, p. 125]. Clearly $|\phi(e^{it})| < 1$ almost everywhere and $\log(1 - |\phi(e^{it})|) \in L^1$. Let $P_z(t)$ for $|z| < 1$ denote $\operatorname{Re}((e^{it}+z)/(e^{it}-z))$. Since $\phi \in H^1$ we have $\phi(e^{it}) \in L^1$ and for $|z| < 1$, $\phi(z) = (1/2\pi) \int_0^{2\pi} P_z(t)\phi(e^{it})\, dt$ [**3**, p. 34]. It follows that

$$1 - |\phi(z)| \geq \frac{1}{2\pi} \int_0^{2\pi} P_z(t)(1 - |\phi(e^{it})|)\, dt. \tag{1}$$

Applying Jensen's inequality we obtain from (1)

$$\log(1 - |\phi(z)|) \geq \frac{1}{2\pi} \int_0^{2\pi} P_z(t) \log(1 - |\phi(e^{it})|)\, dt. \tag{2}$$

Hence we deduce from (2) that

$$\exp\left(\frac{1}{2\pi} \int_0^{2\pi} P_z(t) \log(1 - |\phi(e^{it})|)^2\, dt\right) \leq (1 - |\phi(z)|)^2. \tag{3}$$

Now define

$$g(z) = \frac{z}{16} \exp \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log(1 - |\phi(e^{it})|)^2\, dt.$$

Note that $g(0) = 0$, $g \not\equiv 0$, and g is analytic in Δ. We also have

$$|g(z)| < \frac{1}{16} \exp\left(\frac{1}{2\pi} \int_0^{2\pi} P_z(t) \log(1 - |\phi(e^{it})|)^2\, dt\right)$$

and so this fact and (3) imply

$$|g(z)| < \tfrac{1}{16}(1 - |\phi(z)|)^2. \tag{4}$$

Since F is univalent, it follows from [**7**, p. 22] that

$$\tfrac{1}{4}(1 - |z|^2)|F'(z)| \leq \lambda(F(z), \partial D) \leq (1 - |z|^2)|F'(z)| \tag{5}$$

for all z in Δ. We may assume, without loss of generality, that $F \in S$ and so it is known that

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |F'(z)| \tag{6}$$

for all z in Δ [**7**, p. 21]. It follows from (5) and (6) that

$$\tfrac{1}{16}(1 - |z|)^2 \leq \lambda(F(z), \partial D). \tag{7}$$

Therefore $\frac{1}{16}(1-|\phi(z)|)^2 \leq \lambda(F(\phi(z)), \partial D) = \lambda(f(z), \partial D)$. Hence, by (4) and (7) we have $|g(z)| < \lambda(f(z), \partial D)$ for all z in Δ. We conclude that $f(z) \pm g(z) \in F(\Delta)$ for all z in Δ. Since F is univalent, $f(0) = F(0)$, and $g(0) = 0$, we have $f \pm g \in s(F)$ and $g \equiv 0$. This contradicts the assumption that $f \in Es(F)$.

Hence $\phi \in EB_0$ and the proof is complete.

REMARK. The conjecture made by the first author in [**1**] is now proved.

THEOREM 2. *If F is a univalent function analytic in Δ, $f = F \circ \phi \in Es(F)$, and F' is in the Nevanlinna class, then*

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})), \partial D)\, dt = -\infty. \tag{8}$$

PROOF. Theorem 1 implies $F \circ \phi \in EB_0$ and so we have $\int_0^{2\pi} \log(1-|\phi(e^{it})|)\, dt = -\infty$ [**3**, p. 125]. It follows from this fact and the assumption that F' is in the Nevanlinna class that (8) holds [**1**, p. 440].

REMARK. This result was previously known under the assumptions that $F(\Delta)$ is a Jordan domain and $\phi \in EB_0$ [**5**, pp. 143–144]. We conjecture that (8) holds in general. The following theorem is indirect evidence for this conjecture.

THEOREM 3. *If F is a univalent function analytic in Δ and $f = F \circ \phi \in Es(F)$, then*

$$\inf_r \int_0^{2\pi} \log \lambda(F(re^{it}), \partial D)\, dt = -\infty. \tag{9}$$

PROOF. We first prove that if $\phi \in B_0$, then

$$\inf_r \int_0^{2\pi} \log(1-|\phi(re^{it})|^2)\, dt = \int_0^{2\pi} \log(1-|\phi(e^{it})|^2)\, dt. \tag{10}$$

It is easy to verify by a computation that $-\log(1-|z|^2)$ is a C^∞ subharmonic function in Δ. It follows easily by another computation that $-\log(1-|\phi(z)|^2)$ is subharmonic in Δ and so $-\int_0^{2\pi} \log(1-|\phi(re^{it})|^2)\, dt$ is an increasing function of r [**3**, p. 9].

First consider the case $\int_0^{2\pi} \log(1-|\phi(e^{it})|^2)\, dt = -\infty$. Then by Fatou's lemma we have

$$+\infty = -\int_0^{2\pi} \log(1-|\phi(e^{it})|^2)\, dt \leq \varliminf_{r\to 1} -\int_0^{2\pi} \log(1-|\phi(re^{it})|^2)\, dt. \tag{11}$$

Since $-\int_0^{2\pi} \log(1-|\phi(re^{it})|^2)\, dt$ is an increasing function of r, we have

$$\varliminf_{r\to 1} -\int_0^{2\pi} \log(1-|\phi(re^{it})|^2)\, dt = \sup_r -\int_0^{2\pi} \log(1-|\phi(re^{it})|^2)\, dt = +\infty.$$

It follows that $\inf_r \int_0^{2\pi} \log(1-|\phi(re^{it})|^2)\, dt = -\infty$ and (10) holds in this case.

Now suppose $\int_0^{2\pi} \log(1-|\phi(e^{it})|^2)\, dt > -\infty$. We have $\log(1-|\phi(e^{it})|^2) \in L^1$, and it follows by the type of arguments used in the proof of Theorem 1 that

$$-\log(1-|\phi(re^{i\theta})|^2) \leq \frac{-1}{2\pi} \int_0^{2\pi} P_z(t) \log(1-|\phi(e^{it})|^2)\, dt. \tag{12}$$

Hence by integrating both sides of (12), using Fubini's theorem and the fact that $(1/2\pi)\int_0^{2\pi} P_z(t)\,d\theta = 1$ we have

$$-\int_0^{2\pi} \log(1 - |\phi(re^{it})|^2)\,dt \le -\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\,dt. \tag{13}$$

It follows that

$$\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\,dt \le \inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2)\,dt. \tag{14}$$

A direct application of Fatou's lemma gives

$$-\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\,dt \le \varliminf_{r\to 1} -\int_0^{2\pi} \log(1 - |\phi(re^{it})|^2)\,dt. \tag{15}$$

As we have seen in the preceding argument, the expression on the right-hand side of (15) is equal to $-\inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2)\,dt$ and so we have

$$\inf_r \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2)\,dt \le \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\,dt. \tag{16}$$

It follows from (14) and (16) that (10) holds in this case.

To complete the proof of the theorem we may assume without loss of generality that $F \in S$. Now suppose $f = F\circ\phi \in Es(F)$. Then by Theorem 1, $\phi \in EB_0$. Since $F'(z) \ne 0$ in Δ, we have $\log|F'(z)|$ and $\log|F'(\phi(z))|$ harmonic in Δ. It follows that

$$\int_0^{2\pi} \log|F'(\phi(re^{it}))|\,dt = 2\pi\log|F'(\phi(0))| = 2\pi\log|F'(0)| = 0$$

since $F'(0) = 1$. It follows from this fact and (5) that

$$\int_0^{2\pi} \log\lambda(F(\phi(re^{it})), \partial D)\,dt \le \int_0^{2\pi} \log(1 - |\phi(re^{it})|^2)\,dt. \tag{17}$$

It follows from (10), (17) and the fact that $\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\,dt = -\infty$ when $\phi \in EB_0$ that (9) holds and the proof is complete.

The next two theorems are technical results needed for the proof of Theorem 6.

THEOREM 4. *If F is a bounded univalent function analytic in Δ, $\phi \in B_0$, $|\phi(e^{it})| < 1$ for almost all t, then*

$$\int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})e^{i\theta})|\,dt < +\infty \tag{18}$$

for almost all θ.

PROOF. It is easy to see that the integrand in (18) is a measurable function of (t,θ) on $[0,2\pi]\times[0,2\pi]$ since it is almost everywhere the limit of continuous functions on $[0,2\pi]\times[0,2\pi]$. Since F is univalent and bounded, it follows easily from Theorem 5.2 in [**7**, p. 129] that there exists a constant B such that

$$\int_0^{2\pi} |F'(re^{i\theta})|\,d\theta \le \frac{B}{(1-r)^{1/2-1/320}} \tag{19}$$

for all $r < 1$. Since $|\phi(e^{it})| < 1$ for almost all t, (19) implies

$$\int_0^{2\pi} |F'(\phi(e^{it})e^{i\theta})|\,d\theta \le \frac{B}{(1-|\phi(e^{it})|)^{1/2-1/320}} \tag{20}$$

for almost all t. It follows from (20) that

$$\int_0^{2\pi} (1-|\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})e^{i\theta})|\,d\theta \le B \tag{21}$$

for almost all t. Hence (21) implies that

$$0 \le \int_0^{2\pi}\left(\int_0^{2\pi} (1-|\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})e^{i\theta})|\,d\theta\right) dt < +\infty. \tag{22}$$

The Tonelli-Hobson theorem and (22) imply that

$$0 \le \int_0^{2\pi}\left(\int_0^{2\pi} (1-|\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})e^{i\theta})|\,dt\right) d\theta < +\infty. \tag{23}$$

It follows from (23) that (18) holds and the proof is complete.

REMARK. The set of θ of measure 2π on which (18) holds depends on ϕ. Also the exponent $1/2$ in (18) is not the smallest possible. Clearly $1/2 - 1/320$ will also suffice.

THEOREM 5. *If F is a univalent function analytic in Δ, $\phi \in B_0$, and $|\phi(e^{it})| < 1$ for almost all t, then*

$$\int_0^{2\pi} \log\left[(1-|\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})e^{i\theta})|\right] dt < +\infty \tag{24}$$

for almost all θ.

PROOF. We first show that it is sufficient to consider the case F is univalent and bounded. If F is not bounded, then $F(\Delta)$ cannot be the entire plane, so there is a point b in the complement of $F(\Delta)$. By a simple argument [**8**, pp. 302–303] there is a number c such that $g(z) = [(F(z)-b)^{1/2}+c]^{-1}$ is univalent and bounded in Δ. So $F(z) = b + (1/g(z)-c)^2$ for $|z| < 1$ and g is a bounded univalent function. It follows that we have

$$\begin{aligned}
&\int_0^{2\pi} \log\left[(1-|\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})e^{i\theta})|\right] dt \\
&\quad = 2\pi\log 2 + \int_0^{2\pi} \log|1-cg(\phi(e^{it})e^{i\theta})|\,dt - 3\int_0^{2\pi} \log|g(\phi(e^{it})e^{i\theta})|\,dt \\
&\qquad + \int_0^{2\pi} \log(1-|\phi(e^{it})|^2)^{1/2}|g'(\phi(e^{it})e^{i\theta})|\,dt.
\end{aligned} \tag{25}$$

Since $1-cg(z)$ and $g(z)$ are in H^p for all $p < 1/2$ [**3**, p. 50], the first and second integrals on the right-hand side of (25) are finite [**3**, p. 17]. Since we will prove that the third integral on the right is finite, the proof will be complete.

Now suppose g is a bounded univalent function analytic in Δ. Theorem 4 implies that

$$\int_0^{2\pi} (1-|\phi(e^{it})|^2)^{1/2}|g'(\phi(e^{it})e^{i\theta})|\,dt < +\infty$$

for almost all θ. It is easy to deduce (24) from this fact and this completes the proof.

REMARK. The set of θ of measure 2π on which (24) holds depends on ϕ.

THEOREM 6. *If F is a univalent function analytic in Δ, $\phi \in EB_0$, and $|\phi(e^{it})| < 1$ for almost all t, then*

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D)\, dt = -\infty \tag{26}$$

for almost all θ.

PROOF. Since F is univalent we have from (5) that

$$\lambda(F(\phi(e^{it})e^{i\theta}), \partial D) \leq (1 - |\phi(e^{it})|^2)|F'(\phi(e^{it})e^{i\theta})| \tag{27}$$

for almost all t and all θ. It follows from (27) that

(28)

$$\begin{aligned}\int_0^{2\pi} &\log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D)\, dt \\ &\leq \frac{1}{2}\int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)\, dt + \int_0^{2\pi} \log(1 - |\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})e^{i\theta})|\, dt.\end{aligned}$$

Since $\phi \in EB$, we have $\int_0^{2\pi} \log(1-|\phi(e^{it})|^2)\, dt = -\infty$ [**3**, p. 125] and so (26) follows from this fact, (24), and (28).

REMARK. The set of θ of measure 2π on which (26) holds depends on ϕ.

COROLLARY. *If F is a univalent function analytic in Δ, $\phi \in B_0$, and $F \circ \phi \in Es(F)$, then*

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})e^{i\theta}), \partial D)\, dt = -\infty \tag{29}$$

for almost all θ.

PROOF. This follows directly from Theorem 1 and Theorem 6.

Our next result generalizes the previous theorem.

THEOREM 7. *If F is a univalent function analytic in Δ, $\phi \in EB_0$, $|\phi(e^{it})| < 1$ for almost all t, and ψ is an inner function such that $|\psi(0)| \neq 1$, then*

$$\int_0^{2\pi} \log \lambda(F(\phi(e^{it})\psi(e^{i\theta})), \partial D)\, dt = -\infty \tag{30}$$

for almost all θ.

PROOF. We just prove that under the assumption that F is a bounded univalent function in Δ then

$$\int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})\psi(e^{i\theta}))|\, dt < +\infty \tag{31}$$

for almost all θ. The proof of (30) follows from (31) in the way that Theorem 6 was deduced from Theorems 4 and 5. We note that Fatou's lemma gives

$$\begin{aligned}\int_0^{2\pi} &(1 - |\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})\psi(e^{i\theta}))|\, d\theta \\ &\leq \varliminf_{r\to 1} \int_0^{2\pi} (1 - |\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})\psi(re^{i\theta}))|\, d\theta.\end{aligned} \tag{32}$$

Littlewood's inequality [3, pp. 10–11; 6, p. 243] can easily be generalized so that we have

$$\int_0^{2\pi} |F'(\phi(e^{it})\psi(re^{i\theta}))|\, d\theta \le \frac{r+|\psi(0)|}{r-|\psi(0)|}\int_0^{2\pi} |F'(\phi(e^{it})re^{i\theta})|\, d\theta \tag{33}$$

whenever $|\psi(0)| < r$. Since F is a bounded univalent function, (19) and (13) imply that when $|\psi(0)| < r$,

$$\int_0^{2\pi} |F'(\phi(e^{it})\psi(re^{i\theta}))|\, d\theta \le \frac{r+|\psi(0)|}{r-|\psi(0)|}\frac{B}{(1-|\phi(e^{it})|r)^{1/2-1/320}}. \tag{34}$$

It follows from (32) and (34) that we have

$$\int_0^{2\pi} (1-|\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})\psi(e^{i\theta}))|\, d\theta \le \frac{1+|\psi(0)|}{1-|\psi(0)|}B. \tag{35}$$

We infer from (35) that

$$\int_0^{2\pi}\left(\int_0^{2\pi} (1-|\phi(e^{it})|^2)^{1/2}|F'(\phi(e^{it})\psi(e^{i\theta}))|\, d\theta\right) dt < +\infty. \tag{36}$$

Hence (31) follows from (36) and the Tonelli-Hobson theorem.

REMARK. The set of θ of measure 2π on which (30) holds depends on ϕ and ψ.

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